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LETTER TO THE EDITOR

q-supersymmetric generalization of von Neumann's theorem

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Abstract. Assuming that there exist operators which form an irreducible representation of the q-superoscillator algebra, it is proved that any two such representations are equivalent, related by a uniquely determined superunitary transformation. This provides a q-supersymmetric generalization of the well known uniqueness theorem of von Neumann for any finite number of degrees of freedom.

In the last few years quantum deformations of Lie groups and Lie algebras have found several applications in mathematics and theoretical physics (see e.g. [1–3]). These deformations have been subsequently extended to supergroups and superalgebras [4–6]. In particular, the bosonic and fermionic q-oscillators [7, 8, 6] have been used for the realization of different quantum Lie algebras [7–9] and quantum superalgebras [6, 10, 11].

A natural question then arises, concerning the relation between different irreducible representations of the q-deformed algebras. It is well known that in the case of the classical bosonic and fermionic Heisenberg algebras for harmonic oscillators, this problem is solved by the von Neumann's theorem (see, e.g. [12, 13]), which states that irreducible representations of the bosonic (fermionic) algebra are unitarily equivalent to each other. Similar results also hold for irreducible operator representations of Lie superalgebras (of oscillators) [14]. Recently, it was proved that the analogue of von Neumann's theorem is also valid in the case of q-oscillator algebras [15]. In the latter case, the unitary irreducible representations are non-unique as the algebra has a non-trivial central element [16, 17].

In this letter we shall extend the results of [14] and formulate a quantum supersymmetric generalization of von Neumann's theorem for irreducible representations of q-deformed superalgebras. We start from a supercovariant system of q-oscillators [11], which are covariant under the coaction of a supergroup, $SU_q(n|m)$. The latter present the extension of the covariant system of q-oscillators proposed in [18, 19]. Assuming suitable domain properties as in [14], we prove that any two irreducible representations of the q-deformed superalgebra are connected by a unique *superunitary* transformation+. A similar result is

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+ We stress, however, that if the domain assumption is violated then one can eventually obtain non-equivalent representations. This problem remains open.

also proved to be valid for any finite number, n, of bosonic and fermionic independent q-oscillators. We present the explicit form of the superunitary transformation operator for the cases $n = 1$ and 2.

We start by recalling the classical von Neumann's theorem [12, 13]. Let b, b^+ and b', b'^+ be two irreducible representations of the Heisenberg algebra,

$$bb^+ - b^+b = 1 \tag{1}$$

in the Hilbert spaces H and H', respectively, and assume that there exist vectors $|0>$ in H and $|0'>$ in H', such that $b|0>= 0$, $b'|0'>= 0$. Then there exists a unitary operator U such that

$$\begin{aligned} &b' = UbU^+ \qquad b'^+ = Ub'U^+ \\ &UU^+ = U^+U = 1\,. \end{aligned} \tag{2}$$

A similar theorem also holds in the case of the fermionic algebra [12, 13],

$$cc^+ + c^+c = 1\,. \tag{3}$$

The situation is different for the q-deformed bosonic oscillator algebra [7, 8, 6]

$$\begin{aligned} &aa^+ - qa^+a = q^{-N} \\ &[N, a] = -a \qquad [N, a^+] = a^+ \end{aligned} \tag{4}$$

due to the existence of a non-trivial central element of the algebra [16, 17]. The unitary representations of (4) exist for q positive [15]. Notice, however, that there exist two distinct classes of unitary irreducible representations [16] (see also [20]), one well-behaved for $q \to 1$ and the other singular in the limit $q \to 1$. In the case of well-behaved representations, the Fock representation of (4) can be realized in terms of the non-deformed oscillators (1) as [21]

$$a = \varphi(N)b \qquad a^+ = b^+\varphi^+(N) \tag{5}$$

where

$$N = b^+b \qquad \varphi(N) = \sqrt{\frac{[N+1]}{N+1}} \tag{6}$$

and $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. (For the exceptional values of q being the mth root of unity, $q = \mathrm{e}^{\pm i\pi/m}$, $[m] = 0$ and thus $\varphi(m-1) = 0$. In such a case the Hilbert space becomes finite m-dimensional). The equivalence of such representations† then follows from von Neumann's theorem for usual oscillators:

Let b, b^+ and b', b'^+ satisfy all the conditions of von Neumann's theorem and let us define the operators a, a^+ and a', a'^+ as

$$\begin{aligned} a &= \varphi(N)b \qquad a^+ = b^+\varphi^+(N) \\ a' &= \varphi(N')b' \quad\; a'^+ = b'^+\varphi^+(N') \end{aligned} \tag{7}$$

where

$$N = b^+b \qquad N' = b'^+b' \tag{8}$$

† For the general case of $\varphi_C(N) = \sqrt{\frac{[N+1]_C}{N+1}}$, $[N]_C \equiv [N] + Cq^N$, with the central element C real, while the irreducible representations with the same C are equivalent; they are not unitary equivalent for different values of C [16, 17].

are the number operators and φ is a well-behaved function.

Then for the representations a, a^+ and a', a'^+ in some Hilbert (sub)spaces H_a and H'_a ($H_a \subseteq H$, $H'_a \subseteq H'$), respectively, von Neumann's theorem also holds, i.e. a, a^+ and a', a'^+ are irreducible representations; then there exist vectors $|0>$ in H_a and $|0'>$ in H'_a such that $a|0>=0$, $a'|0'>=0$ and there exists a unitary operator U such that

$$a' = UaU^+ \qquad a'^+ = Ua^+U^+ \qquad N' = UNU^+ . \tag{9}$$

To prove (9) we notice that since b, b^+ and b', b'^+ satisfy the conditions of von Neumann's theorem, it follows that there exists a unitary operator U such that relations (2) hold. From (2) and (8) it follows that

$$N' = b'^+b' = Ub^+U^+UbU^+ = UNU^+$$

and the same relation is also valid for the function $\varphi(N)$,

$$\varphi(N') = U\varphi(N)U^+ .$$

Now, from the definition of the operators a and a^+ (equations (7)) we have

$$UaU^+ = U\varphi(N)bU^+ = U\varphi(N)U^+UbU^+ = \varphi(N')b' = a'$$

and, similarly,

$$Ua^+U^+ = a'^+ .$$

Relations (9) are thus proved.

It is also clear that in the spaces H_a and H'_a there exist vectors $|0>$ and $|0'>$ such that $a|0>=0$, $a'|0'>=0$. Finally, a, a^+ and a', a'^+ are two irreducible representations in the Hilbert spaces H_a, H'_a, respectively. This follows from the definitions (7) and the fact that b, b^+ and b', b'^+ are irreducible representations in H, H'.

A similar statement is also valid in the case of the q-deformed fermionic algebra [6]

$$ff^+ + qf^+f = q^M$$
$$[M, f] = -f \qquad [M, f^+] = f^+$$

which can be obtained from the algebra (3) by means of the change of operators

$$f = q^{M/2}c \qquad f^+ = c^+q^{M/2} .$$

Let us remark that the above statements were proved in [15] by following a proof along the same lines as the original von Neumann's theorem for usual (non-deformed) oscillators and valid also for generic values of q including the exceptional values of the mth root of unity. The same theorem is also valid in the case of the other q-fermionic algebra [22] given by the commutation relations

$$ff^+ + qf^+f = q^{-M}$$
$$[M, f] = -f \qquad [M, f^+] = f^+ .$$

In fact, we can obtain this set from the bosonic algebra (1) by means of the transformation

$$f = \sqrt{\frac{[M+1]^f}{M+1}}b \qquad f^+ = b^+\sqrt{\frac{[M+1]^f}{M+1}}$$

where $[n]^f = (q^{-n} - (-1)^n q^n)/(q + q^{-1})$, since in the Fock space we have the relations $f^+f = [M]^f$, $ff^+ = [M+1]^f$.

We now formulate our main result, namely, we prove that an irreducible representation of the q-deformed superalgebras is unique, up to a superunitary transformation, in the following sense:

Theorem (i)

Let $Z = \{B, B^+, F, F^+\}$ be an irreducible operator family, which satisfies the q-deformed superalgebra [11]:

$$BF = qFB \qquad B^+F^+ = q^{-1}F^+B^+ \tag{10a}$$

$$BF^+ = q^{-1}F^+B \qquad B^+F = qFB^+ \tag{10b}$$

$$F^2 = (F^+)^2 = 0 \tag{10c}$$

$$BB^+ - q^{-2}B^+B = 1 + (q^{-2} - 1)F^+F \tag{10d}$$

$$FF^+ + F^+F = 1 \tag{10e}$$

where q is a real number, F and F^+ are bounded operators on the separable Hilbert space H and B, B^+ are densely defined closed operators in H. Let θ be a Grassmann variable, such that

$$\{\theta, F\} = \{\theta, F^+\} = \theta^2 = 0 \tag{11a}$$

$$[\theta, B] = [\theta, B^+] = 0\,. \tag{11b}$$

Let D and G be densely defined closed linear operators, defined on a suitable domain [14]

$$\begin{aligned} B' &= B + \theta D \qquad B'^+ = B^+ - \theta D^+ \\ F' &= F + \theta G \qquad F'^+ = F^+ + \theta G^+ \end{aligned} \tag{12}$$

$$\begin{aligned} G &= G_{00}(B, B^+) + G_{11}(B, B^+)F^+F \\ D &= D_{10}(B, B^+)F^+ + D_{01}(B, B^+)F \end{aligned} \tag{13}$$

where G and D are assumed to be even and odd Grassmann elements, respectively. Assume that the operator family $Z' = \{B', B'^+, F', F'^+\}$ also fulfills the algebra (10) on a suitable domain of definition.

Then, under the above conditions, there exists a uniquely determined self-adjoint odd operator A, such that

$$G = \{A, F\} \qquad D = [A, B] \tag{14}$$

$$A = F^+G_{00}(B, B^+) \; + \; G_{00}^+(B, B^+)F = A^+ \tag{15}$$

and the transformation (12) is implemented by the superunitary operator $e^{\theta A}$ such that

$$\begin{aligned} e^{\theta A}Be^{-\theta A} &= B + \theta[A, B] = B' \\ e^{\theta A}Fe^{-\theta A} &= F + \theta\{A, F\} = F'\,. \end{aligned} \tag{16}$$

Under the conditions of the above theorem, we have

$$G_{11}(B, B^+) = G_{00}(qB, qB^+) - G_{00}(B, B^+) \tag{17a}$$

$$\begin{aligned} D_{10}(B, B^+) &= qG_{00}(qB, qB^+)B - BG_{00}(qB, qB^+) \\ D_{01}(B, B^+) &= q^{-1}G_{00}^+(B, B^+)B - BG_{00}^+(B, B^+)\,. \end{aligned} \tag{17b}$$

The transformation $e^{\theta A} = 1 + \theta A$ is called *superunitary* if A is odd and self-adjoint. In particular, from this definition it follows that $\{\theta, A\} = 0$ and the latter is fulfilled only when θ satisfies the commutation relations (11).

Remark:

The Grassmann variable θ, being on the q-plane of the oscillators given in (10), can in general have the q-commutation relations with the elements of the algebra as:

$$\theta_q F + pF\theta_q = 0 \qquad \theta_q F^+ + p^{-1}F^+\theta_q = 0$$
$$\theta_q B - rB\theta_q = 0 \qquad \theta_q B^+ - r^{-1}B^+\theta_q = 0$$

with p and r real numbers†. If however, we use the above q-commutation relations with p and r, instead of commutation relations (11), from the requirement of oddness of operator A and the superunitarity condition $(\mathrm{e}^{\theta_q A})^+ = \mathrm{e}^{-\theta_q A}$, i.e. $A^+\theta_q = -\theta_q A$, one arrives at the following 'q-self adjointness' condition,

$$A^+(B, B^+, F, F^+) = A(rB, r^{-1}B^+, pF, p^{-1}F^+)$$

instead of the usual one. The latter restriction seems rather unnatural, since a usual self-adjoint operator A on the original algebra of oscillators (10) now acquires restrictions by the inclusion of an additional auxiliary Grassmann element θ_q. Thus for our purpose we can restrict ourselves to the case $p = r = 1$, i.e. to the commutation relations (11).

Proof of the theorem:

The proof is actually quite similar to the one given in [14] for the classical Lie superalgebras.

From the relations $F^2 = F'^2 = 0$ and using (11a) we have, according to (12), $F'^2 = \theta[G, F] = 0$, from which we obtain

$$[G, F] = 0\,. \tag{18}$$

Relations (10a), (10b) imply that

$$\begin{aligned} g(B, B^+)F &= Fg(qB, qB^+) \\ g(B, B^+)F^+ &= F^+g(q^{-1}B, q^{-1}B^+) \end{aligned} \tag{19}$$

for any function $g(B, B^+)$. With the ansatz (13) and using (10e) and (19) we find

$$[G, F] = \{G_{00}(B, B^+) - G_{00}(q^{-1}B, q^{-1}B^+) - G_{11}(q^{-1}B, q^{-1}B^+)\}F\,. \tag{20}$$

Comparing (18) and (20) we get

$$G_{11}(B, B^+) = G_{00}(qB, qB^+) - G_{00}(B, B^+) \tag{21}$$

and substituting it into (13) we obtain

$$G = G_{00}(B, B^+)FF^+ + G_{00}(qB, qB^+)F^+F\,. \tag{22}$$

Let us now take G, in the required form (14), as

$$G = \{A, F\} \tag{23}$$

† These q-commutation relations, because of associativity, are of course compatible with the supersymmetric q-Jacobi identities

$$[[A, B\}_{(q_3, q_3^{-1})}, C\}_{(\frac{q_1}{q_2}, \frac{q_2}{q_1})} + (-1)^{\eta_A(\eta_B+\eta_C)}[[B, C\}_{(q_1, q_1^{-1})}, A\}_{(\frac{q_2}{q_3}, \frac{q_3}{q_2})}$$
$$+(-1)^{\eta_C(\eta_A+\eta_B)}[[C, A\}_{(q_2, q_2^{-1})}, B\}_{(\frac{q_3}{q_1}, \frac{q_1}{q_3})} = 0$$

where $\eta_Z = 1$ if Z is odd, $\eta_Z = 0$ if Z is even and $[A, B\}_{(p,q)} \equiv pAB \mp qBA$, where we take the plus sign when both A and B are odd, otherwise we take the minus sign. The above expression represents the most general form of the supersymmetric q-Jacobi identities, which includes three arbitrary complex parameters q_1, q_2 and q_3.

where A is odd. Assuming that A is self-adjoint, i.e. $A^+ = A$, we can write

$$A = F^+\alpha(B, B^+) + \alpha^+(B, B^+)F \tag{24}$$

and we have from (23)

$$G = \alpha(qB, qB^+)F^+F + \alpha(B, B^+)FF^+. \tag{25}$$

Comparing (22) and (25), we find

$$\alpha(B, B^+) = G_{00}(B, B^+)$$

and by substituting it back into (24), the relation (15) is obtained.

Now to find D, we use relations (10a) and (11). We have

$$B'F' - qF'B' = \theta\{DF + BG - qGB + qFD\} = 0\,. \tag{26}$$

Substituting (23) into (26) and using (10a), we will obtain then that

$$\{D - [A, B]\}F + qF\{D - [A, B]\} = 0\,. \tag{27a}$$

Similarly, from (10b), (11) and $G^+ = \{A, F^+\}$ we obtain

$$\{D - [A, B]\}F^+ + q^{-1}F^+\{D - [A, B]\} = 0\,. \tag{27b}$$

Thus, from equations (27) we conclude that

$$D = [A, B]\,. \tag{28}$$

Finally, the operator $e^{\theta A}$ is superunitary since A is odd and self-adjoint (see equation (24)). Moreover, we have

$$e^{\theta A}Be^{-\theta A} = (1 + \theta A)B(1 - \theta A) = B + \theta[A, B] = B'$$
$$e^{\theta A}Fe^{-\theta A} = (1 + \theta A)F(1 - \theta A) = F + \theta\{A, F\} = F'$$

i.e. relations (16) hold. The proof of the relations (17) is straightforward: equation (17a) was already proved (see equation (21)) and relations (17b) follow from (13), (14), (15) and (19). This completes the proof. When $q = 1$, we reproduce the results of [14].

Let us remark that a pair of one bosonic, b, b^+, and one fermionic, f, f^+, independent q-oscillators (which satisfy the relations (30)), can also be introduced by means of the transformation [11]

$$\begin{array}{lll} b = q^{\frac{N}{2}+M}B & f = q^{\frac{M}{2}}F & \\ [N, B] = -B & [N, B^+] = B^+ & N^+ = N \\ [M, F] = -F & [M, F^+] = F^+ & M^+ = M \end{array} \tag{29}$$

where B and F are the elements of supercovariant algebra (10). In this case, the uniqueness (up to a superunitary transformation) of any irreducible representation of b, b^+, N, f, f^+, M is given by the following

Theorem (ii)

Let $Z = \{b, b^+, N, f, f^+, M\}$ be an irreducible operator family, which satisfies the q-oscillator algebra

$$[b, f] = 0 \qquad [b, f^+] = 0 \tag{30a}$$
$$f^2 = (f^+)^2 = 0 \tag{30b}$$
$$bb^+ - q^{-1}b^+b = q^N \tag{30c}$$
$$ff^+ + qf^+f = q^M \tag{30d}$$
$$[N, b] = -b \qquad [N, b^+] = b^+ \tag{30e}$$
$$[M, f] = -f \qquad [M, f^+] = f^+ \tag{30f}$$

on a suitable domain of definition; q is real. Let θ be a Grassmann variable

$$\begin{aligned}&\{\theta, f\} = \{\theta, f^+\} = \theta^2 = 0\\&[\theta, b] = [\theta, b^+] = 0\,.\end{aligned} \tag{31}$$

Define

$$\begin{aligned}&b' = b + \theta D \quad f' = f + \theta G\\&D = D_{10}(b, b^+, M) f^+ + D_{01}(b, b^+, M) f\\&G = G_{00}(b, b^+, M) + G_{11}(b, b^+, M) f^+ f\end{aligned} \tag{32}$$

where D and G, the even and odd Grassmann elements, respectively, have the most general form as in (32). Assume that the operator family $Z' = \{b', b'^+, N', f', f'^+, M'\}$ also fulfills the algebra (30) on a suitable domain of definition.

Then there exists a uniquely determined self-adjoint odd operator A, such that one can write

$$G = \{A, f\} \qquad D = [A, b] \tag{33}$$

$$A = f^+\alpha(b, b^+, M) + \alpha^+(b, b^+, M) f = A^+ \tag{34}$$

and the transformation (32) is implemented by the superunitary operator $e^{\theta A}$ such that

$$\begin{aligned}&e^{\theta A} b e^{-\theta A} = b'\\&e^{\theta A} f e^{-\theta A} = f'\,.\end{aligned} \tag{35}$$

Under the conditions of the above theorem (ii), we have

$$\begin{aligned}&G_{00}(b, b^+, M) = \alpha(b, b^+, M) q^M\\&G_{11}(b, b^+, M) = \alpha(b, b^+, M-1) - q\alpha(b, b^+, M)\end{aligned} \tag{36a}$$

and

$$D_{10}(b, b^+, M) = [\alpha(b, b^+, M-1), b] \quad D_{01} = [\alpha^+(b, b^+, M), b]. \tag{36b}$$

The proof is similar to the one of theorem (i) and need not be given.

Let us notice that from the theorem (ii), it also follows that

$$\begin{aligned}&e^{\theta A} N e^{-\theta A} = N + \theta[A, N] = N'\\&e^{\theta A} M e^{-\theta A} = M + \theta[A, M] = M'\,.\end{aligned} \tag{37}$$

Indeed, relations (35) imply that for any functions $\varphi(bb^+, b^+b)$ and $\psi(ff^+, f^+f)$, one has

$$\begin{aligned}&e^{\theta A}\varphi(bb^+, b^+b)\, e^{-\theta A} = \varphi(b'b'^+, b'^+b')\\&e^{\theta A}\psi(ff^+, f^+f)\, e^{-\theta A} = \psi(f'f'^+, f'^+f')\end{aligned}$$

$\varphi = N$ and $\psi = M$ are just particular cases of these functions (see equations (30c) and (30d), respectively).

Here we would like to mention that there exists a relation between the transformations of theorems (i) and (ii). Indeed, if one takes in (34)

$$\alpha(b, b^+, M) = q^{-M/2} G_{00}(B, B^+) \tag{38}$$

where $B = q^{-\frac{1}{2}N-M} b$, $F = q^{-M/2} f$ according to (29), it is straighforward to show that the superunitary transformation generated by (34), (35) with α given in (38), corresponds to the superunitary transformation (15), (16) of theorem (i), and therefore equations (17) are satisfied.

Also it will be interesting to find a direct relation between the superunitary transformations of the usual (undeformed) and of the q-deformed cases obtained here, in the same way as such a relation exists for the non-supersymmetric q-oscillators treated in section 2.

To summarize, we have shown that the irreducible representations of the q-superoscillator algebra are equivalent and are related by a unique superunitary transformation. The q-supersymmetric generalization of von Neumann's theorem, presented above ($N = 1$ supersymmetry), is for only one bosonic and one fermionic degrees of freedom. Our results can be extended to the supersymmetric case with any number of bosonic and fermionic degrees of freedom. The theorem now can be formulated as follows:

Theorem (finite degrees of freedom):

Let $Z = \{b_k, b_k^+, N_k, f_k, f_k^+, M_k;\ k = 1, ..., n\}$ be an irreducible operator set, which satisfies the q-oscillator algebra

$$\begin{aligned}
&b_k b_k^+ - q^{-1} b_k^+ b_k = q^{N_k}\\
&f_k f_k^+ + q\ f_k^+ f_k = q^{M_k}\\
&[N_k, b_k] = -b_k \qquad [N_k, b_k^+] = b_k^+\\
&[M_k, f_k] = -f_k \qquad [M_k, f_k^+] = f_k^+
\end{aligned} \tag{39}$$

with all the other (anti)commutation relations vanishing (i.e. independent system of q-oscillators); q is real.

Assume now that another set of operators $Z' = \{b'_k, b'^+_k, N'_k, f'_k, f'^+_k, M'_k;\ k = 1, ...n\}$ also satisfies the same algebra (39). Then the two sets Z and Z' are equivalent, related by a unique superunitary transformation such that

$$b'_k = U b_k U^+ \qquad f'_k = U f_k U^+ \qquad U = \mathrm{e}^{\theta A} \tag{40}$$

with the same relation between the remaining elements of the two sets. In (40) θ is a Grassmann variable satisfying the (anti)commutation relations (31) for all the b_k, b_k^+, f_k, f_k^+ and A is a self-adjoint odd operator.

The proof of the theorem can be most easily performed by the method of induction in the number of degrees of freedom, n. In addition to the explicit form of the superunitary operator for $n = 1$ given in theorem (ii), we present below explicitly the formulae for the case $n = 2$.

Define

$$b'_k = b_k + \theta D_k \qquad f'_k = f_k + \theta G_k \qquad k = 1, 2 \tag{41}$$

where G_k have the most general form

$$\begin{aligned}
G_k = G_k^0 + G_k^1 f_1^+ f_1 + G_k^2 f_2^+ f_2 + G_k^{12} f_1^+ f_1 f_2^+ f_2\\
+ H_k f_1 f_2 + H'_k f_1^+ f_2^+ + K_k f_1 f_2^+ + K'_k f_1^+ f_2
\end{aligned} \tag{42}$$

all the coefficients in (42) can depend on $b_\ell, b_\ell^+, N_\ell, M_\ell$ $(\ell = 1, 2)$. From the anticommutation relations $\{f'_k, f'_\ell\} = \{f'^+_k, f'^+_\ell\} = \{f'_1, f'^+_2\} = \{f'^+_1, f'_2\} = 0$, we obtain that $H'_k = K'_k = 0$ and that the coefficients $G_k^k, G_k^{12}, H_k, K_k$ $(k = 1, 2)$ can be expressed in terms of $G_1^0, G_2^0, G_2^1, G_1^2$ and their Hermitian conjugates. Now if we write G_k in the form

$G_k = \{A, f_k\}$, then it is straightforward to show that there exists a unique self-adjoint odd operator A given by

$$A = \sum_k f_k^+ G_k^0 q^{-M_k} + \sum_{k \neq \ell} f_k^+ f_\ell^+ f_\ell G_k^\ell q^{-M_k} + \mathrm{HC}$$

and from the commutativity between the b'_k and f'_ℓ, we obtain the required form $D_k = [A, b_k]$.

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